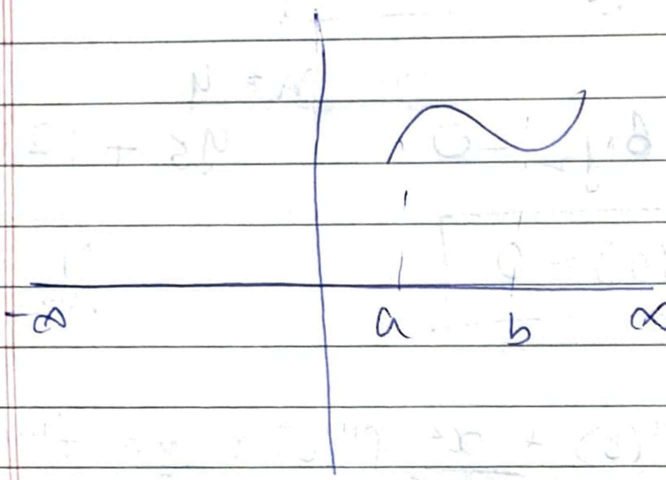
PARTIAL DIFFERENTIATION

~~Geometric & Mathematic Applications~~ of

curves.

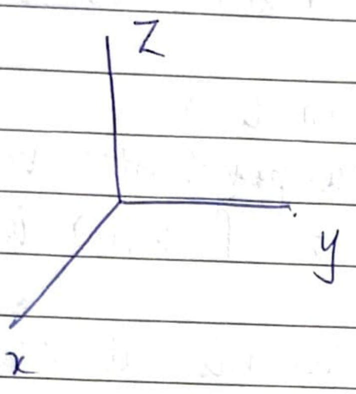


(1) $\frac{dy}{dx}$

(2) $S = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$

(3) $A = \int_a^b y\, dx$

3-D surfaces.

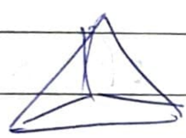


$z = f(x, y)$

paraboloid.

Two kinds of surfaces.

(1) Flat surfaces
- unique normal to every surface, for any point.



tetrahedron

(2) Curved surfaces
- no unique normal for all points on the surface.

- Cross-section of paraboloid, $y = \text{const}$, z varies with x.
- Infinitely many tangents at a point
- Partial derivatives gives slope of tangent at a point in a particular direction

Functions of two or more variables

- (two independent variables)
- If x, y are two independent variables, then the relation $z = f(x,y)$ is called a function of x & y.
- Geometrically, it represents a surface in three dimensions

hyper-surface
- In general, if $x_1, x_2, x_3, \ldots$ are independent variables, then $z = f(x_1, x_2, x_3 \ldots)$ is called a function of $x_1, x_2, x_3 \ldots$ (several variables).

PARTIAL DERIVATIVES

Let $z = f(x,y)$ be a function of 2 independent variables x and y. ~~Then~~,

Then, the first order partial derivative of f with respect to x is denoted by $\frac{\partial f}{\partial x} = \frac{\partial z}{\partial x}$ (or z_x or f_x) and is defined by

∂ - partial
δ - increment
d - total

$$\frac{\partial f}{\partial x} = \lim_{\delta x \to 0} \frac{f(x+\delta x, y) - f(x,y)}{\delta x}$$

provided the limit exists uniquely and finitely.

Similarly, the first order partial derivative of f with respect to y is denoted by $\frac{\partial f}{\partial y} = \frac{\partial z}{\partial y}$ or z_y or f_y and is defined by

$$\frac{\partial f}{\partial y} = \lim_{\delta y \to 0} \frac{f(x, y+\delta y) - f(x,y)}{\delta y}$$

Provided the limit exists uniquely and finitely

$$z = x^3 + y^3 + 3ax^2y + 3axy^2$$

$$\frac{\partial z}{\partial x} = z_x = 3x^2 + 6axy + 3ay^2$$

$$\frac{\partial z}{\partial y} = z_y = 3y^2 + 3ax^2 + 6ax$$

The second order partial derivatives are defined as follows

(1). $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = f_{xx}$

(2) $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = f_{xy}$

(3) $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = f_{yx}$

mixed partial derivatives.

(4) $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = f_{yy}$

Similarly, higher order partial derivatives of f can be defined
In general, there are 2^n n^{th} order ~~derivati~~ partial derivatives of f. (for 2 independent vars)
If there are m independent variables in the base function, ~~&~~ m^n n^{th} order partial derivatives ~~exist~~ of f exist

Problems

1. Verify that $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$ for the following functions:

(a) $z = x^y + y^x$

(b) $z = x^2 \tan^{-1}\left(\frac{y}{x}\right) - y^2 \tan^{-1}\left(\frac{x}{y}\right)$

~~(a) for $u = x^y$~~

~~$\ln u = y \ln x$~~

~~(a)~~ For homogeneous functions, mixed partial derivatives are always equal.
Converse not always true

(a) $z = x^y + y^x$ —— (1)

Diff (1) partially wrt y

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left(x^y + y^x\right)$$

$$= x^y \ln x + x y^{x-1}$$

$$\frac{\partial z}{\partial y} = x^y \ln x + x y^{x-1} \quad \text{—— (2)}$$

Diff. (2) partially wrt x.

$$\frac{\partial^2 z}{\partial x \partial y} = \left(y x^{y-1} \ln x + \frac{x^y}{x}\right) + \left(y^{x-1} + x \cdot y^{x-1} \ln y\right)$$

$$\frac{\partial^2 z}{\partial x \partial y} = yx^{y-1}\ln x + x^{y-1} + y^{x-1} + xy^{x-1}\ln y \quad \text{(3)}$$

Diff. (1) partially w.r.t x.

$$\frac{\partial z}{\partial x} = yx^{y-1} + y^x \ln y \quad \text{--- (4)}$$

Diff. (4) partially w.r.t y.

$$\frac{\partial^2 z}{\partial y \partial x} = x^{y-1} + yx^{y-1}\ln x + xy^{x-1}\ln y + y^{x-1} \quad \text{(5)}$$

We observe that RHS of (3) = RHS of (5)

$$\therefore \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$$

(b) $z = x^2 \tan^{-1}\left(\frac{y}{x}\right) - y^2 \tan^{-1}\left(\frac{x}{y}\right)$ --- (1)

Diff (1) partially w.r.t. x

$$\frac{\partial z}{\partial x} = 2x \tan^{-1}\left(\frac{y}{x}\right) + \frac{x^2}{1+\frac{y^2}{x^2}}\left(\frac{-1}{x^2}\right) \cdot y$$

$$- \frac{y^2}{y}\left(\frac{1}{1+\frac{x^2}{y^2}}\right)$$

$$= \cancel{2x \tan^{-1}\left(\frac{y}{x}\right)}$$

$$\frac{\partial z}{\partial x} = 2x\tan^{-1}\left(\frac{y}{x}\right) - \frac{yx^2}{x^2+y^2} - \frac{y^3}{y^2+x^2}$$

$$\frac{\partial z}{\partial x} = 2x\tan^{-1}\left(\frac{y}{x}\right) - y\frac{\cancel{(x^2+y^2)}}{\cancel{(x^2+y^2)}}$$

$$\frac{\partial z}{\partial x} = 2x\tan^{-1}\left(\frac{y}{x}\right) - y \quad \text{——} \quad (2)$$

~~Diff. (2) partially wrt y~~

$$\cancel{\frac{\partial^2 z}{\partial y\,\partial x} = \frac{2x^2}{1+\frac{x^2}{y^2}}\left(\frac{-1}{y^2}\right) - 1}$$

$$\cancel{= \frac{2x^2 y^2}{y^2+x^2}\left(\frac{-1}{y^2}\right) - 1}$$

$$\frac{\partial^2 z}{\partial y\,\partial x} = \frac{-2x^2}{x^2+y^2} - 1 \longrightarrow (3)$$

Diff (1) partially wrt y

$$\frac{\partial z}{\partial y} = \frac{x^2}{1+\frac{y^2}{x^2}} \times \frac{1}{x} - 2y\tan^{-1}\frac{x}{y}$$

$$- \frac{y^2}{1+\frac{x^2}{y^2}} \cdot x\left(\frac{-1}{y^2}\right)$$

$$\frac{\partial z}{\partial y} = \frac{x^3}{x^2+y^2} - 2y\tan^{-1}\frac{x}{y} + \frac{x\,y^2}{y^2+x^2}$$

$$\frac{\partial z}{\partial y} = x\left(\frac{x^2+y^2}{x^2+y^2}\right) - 2y\tan^{-1}\frac{x}{y}$$

$$\frac{\partial z}{\partial y} = x - 2y\tan^{-1}\frac{x}{y} \longrightarrow (4)$$

Diff (4) wrt x partially

$$\cancel{\frac{\partial^2 z}{\partial x\,\partial y} = 1 - 2\tan^{-1}\frac{x}{y} - 2y\left(\frac{x}{\frac{y^2+x^2}{y^2}}\right)\left(\frac{-1}{y^2}\right)}$$

$$= \cancel{1 - 2\tan^{-1}\frac{x}{y}}$$

$$\frac{\partial^2 z}{\partial x\,\partial y} = 1 - 2y\left(\frac{1}{1+\frac{x^2}{y^2}}\right)\times\frac{1}{y}$$

$$= 1 - \left(\frac{2y^2}{y^2+x^2}\right) = \frac{x^2-y^2}{x^2+y^2} \rightarrow (A)$$

Diff (2) partially wrt y

$$\frac{\partial^2 z}{\partial y\,\partial x} = \frac{2x}{1+\frac{y^2}{x^2}}\times\frac{1}{x} - 1 = \frac{2x^2-x^2-y^2}{x^2+y^2}$$

$$= \frac{x^2-y^2}{x^2+y^2} \longrightarrow (B)$$

i.e. $\dfrac{\partial^2 z}{\partial x\,\partial y} = \dfrac{\partial^2 z}{\partial y\,\partial x}$

2. If $u = e^{xyz}$, find $\frac{\partial^3 u}{\partial x \partial y \partial z}$

$$u = e^{xyz} \quad \text{—(1)}$$

Diff. (1) wrt z partially

$$\frac{\partial u}{\partial z} = xy \cdot e^{xyz} \quad \text{—(2)}$$

Diff (2) partially wrt y

$$\frac{\partial^2 u}{\partial y \partial z} = x \cdot \left(e^{xyz} + yxz e^{xyz}\right)$$

$$\frac{\partial^2 u}{\partial y \partial z} = x e^{xyz} + x^2 yz e^{xyz} \quad \text{—(3)}$$

Diff. (3) partially wrt x

$$\frac{\partial^3 u}{\partial x \partial y \partial z} = e^{xyz} \cdot \cancel{yz} + xyz e^{xyz}$$

$$+ yz\left(2x e^{xyz} + x^2 yz e^{xyz}\right)$$

$$\cancel{= 2xy e^{xyz} + xyz e^{xyz}}$$

$$+ yz e^{xyz} + xy^2 z^2 e^{xyz}$$

$$= e^{xyz} + xyz e^{xyz} + 2xyz e^{xyz} + x^2 y^2 z^2 e^{xyz}$$

$$\frac{\partial^3 u}{\partial x \partial y \partial z} = 3xyz e^{xyz} + x^2y^2z^2 e^{xyz} + e^{xyz}$$

3. If $u = \ln\sqrt{x^2+y^2+z^2}$, show that

$$(x^2+y^2+z^2)\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = \_\_\_$$

$$u = \frac{1}{2}\ln(x^2+y^2+z^2) \quad \text{--- (1)}$$

$\cancel{\frac{\partial u}{\partial x}}$ Diff. (1) partially wrt x.

$$(2) \leftarrow \frac{\partial u}{\partial x} = \frac{1}{\cancel{2}} \times \frac{\cancel{2}x}{(x^2+y^2+z^2)} = \frac{x}{x^2+y^2+z^2}$$

Diff. (1) partially wrt y.

$$(3) \leftarrow \frac{\partial u}{\partial y} = \frac{1}{2} \times \frac{2y}{(x^2+y^2+z^2)} = \frac{y}{x^2+y^2+z^2}$$

Diff. (1) partially wrt z

$$(4) \leftarrow \frac{\partial u}{\partial z} = \frac{z}{x^2+y^2+z^2}$$

Diff. (2) wrt x partially

$$\frac{\partial^2 u}{\partial x^2} = \frac{(1)(x^2+y^2+z^2) - (x)(2x)}{(x^2+y^2+z^2)^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{y^2+z^2-x^2}{(x^2+y^2+z^2)^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{x^2+z^2-y^2}{(x^2+y^2+z^2)^2}$$

$$\frac{\partial^2 u}{\partial z^2} = \frac{x^2+y^2-z^2}{(x^2+y^2+z^2)^2}$$

$$\cancel{\partial \ldots = x^2+y^2+z^2 + x^2+x^2-y^2 + x^2+y^2-z^2}$$

$$\frac{(x^2+y^2+z^2)}{(x^2+y^2+z^2)^2}\left(x^2+y^2-z^2 + x^2+z^2-y^2 + y^2+z^2-x^2\right)$$

LHS $= 1$

4. If $\theta = t^n e^{\frac{-r^2}{4t}}$, what value of n will make $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \theta}{\partial r}\right) = \frac{\partial \theta}{\partial t}$?

To find $\frac{\partial \theta}{\partial t}$ $\qquad \theta = t^n e^{\frac{-r^2}{4t}} \rightarrow (1)$

Diff. (1) ~~wrt~~ partially wrt t.

$$\frac{\partial \theta}{\partial t} = nt^{n-1} e^{\frac{-r^2}{4t}} + t^n\left(\frac{-r^2}{4}\right)e^{\frac{-r^2}{4t}} \cdot \left(\frac{-1}{t^2}\right)$$

$$\frac{\partial \theta}{\partial t} = nt^{n-1} e^{\frac{-r^2}{4t}} + \frac{t^n r^2}{4t^2} e^{\frac{-r^2}{4t}} \rightarrow (2)$$

Diff (1) partially wrt r

$$\frac{\partial \theta}{\partial r} = t^n e^{\frac{-r^2}{4t}} \cdot \left(\frac{-2r}{4t}\right) \longrightarrow (3)$$

$r^2 \times (3)$

$$\frac{r^2 \partial \theta}{\partial r} = -\frac{t^n e^{\frac{-r^2}{4t}} (2r^3)}{4t} \longrightarrow (4)$$

Diff (4) partially wrt r.

$$\frac{\partial}{\partial r}\left(\frac{r^2 \partial \theta}{\partial r}\right) = \frac{-t^n}{4t}\left[e^{\frac{-r^2}{4t}} \cdot \left(\frac{-2r}{4t}\right)(2r^3) + e^{\frac{-r^2}{4t}} \cdot 6r^2\right]$$

$$\frac{\partial}{\partial r}\left(\frac{r^2 \partial \theta}{\partial r}\right) = \frac{-t^n}{4t}\left[e^{\frac{-r^2}{4t}} \cdot \left(\frac{-r^4}{t}\right) + e^{\frac{-r^2}{4t}} \cdot 6r^2\right] \cancel{(5)}$$

~~Dividing (5) by r^2~~

$$\cancel{\frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{r^2 \partial \theta}{\partial r}\right) = \frac{-t^n}{4t}\left[e^{\frac{-r^2}{4t}}\left(\frac{-r^2}{t}\right) + e^{\frac{-r^2}{4t}} \cdot 6\right]}$$

$$\frac{\partial}{\partial r}\left(\frac{r^2 \partial \theta}{\partial r}\right) = \frac{+t^n r^4}{4t^2} e^{\frac{-r^2}{4t}} - \frac{3t^n r^2}{2t} e^{\frac{-r^2}{4t}} \rightarrow (5)$$

Dividing (5) by r^2

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{r^2 \partial \theta}{\partial r}\right) = \frac{t^{n-2} r^2}{4} e^{\frac{-r^2}{4t}} - \frac{3t^{n-1}}{2} e^{\frac{-r^2}{4t}} \text{ --- } (6)$$

Given RHS = LHS

$$\cancel{n t^{n-1} e^{-r^2/4t}} + \frac{t^{n} r^2}{4t^2} \cancel{e^{-r^2/4t}}$$

$$= \frac{t^{n-2} r^2}{4} \cancel{e^{-r^2/4t}} - \frac{3}{2} t^{n-1} \cancel{e^{-r^2/4t}}$$

$$n t^{n-1} + \cancel{\frac{t^{n-2} r^2}{4}} = \cancel{\frac{t^{n-2} r^2}{4}} - \frac{3}{2} t^{n-1}$$

$$\boxed{n = -3/2}$$

~~Redo~~:

~~(b) $z = x^2 \tan^{-1}$~~

5- If $x^x y^y z^z = c$, show that $\frac{\partial^2 z}{\partial x \partial y} = -(x \ln xe)^{-1}$ at $x = y = z$

$x^x y^y z^z = c$

~~$z^z = c x^{-x} y^{-y}$~~

Taking ln on both sides

$$x \ln x + y \ln y + z \ln z = \ln c \rightarrow (1)$$

Differentiating (1) partially wrt y:

$$\frac{y}{y} + \ln y + \frac{z}{z}\frac{\partial z}{\partial y} + \frac{\partial z}{\partial y}\ln z = 0$$

$$1 + \ln y + \frac{\partial z}{\partial y}\left(1 + \ln z\right) = 0$$

$$\frac{\partial z}{\partial y}(1+\ln z) = -1 - \ln y$$

$$\frac{\partial z}{\partial y} = \frac{-(1+\ln y)}{(1+\ln z)} \longrightarrow (2)$$

Differentiating (2) ~~wrt~~ partially wrt x.

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{-\left(\frac{1}{y}\cancel{\frac{\partial y}{\partial x}}^{0}\right)(1+\ln z) + (1+\ln y)\left(\frac{1}{z}\frac{\partial z}{\partial x}\right)}{(1+\ln z)^2}$$

$$(3) \longleftarrow \cancel{\frac{\partial^2 x}{\partial x \partial y} = \frac{(1+\ln y)\left(\frac{1}{z}\frac{\partial z}{\partial x}\right) - \left(\frac{1}{y}\frac{\partial y}{\partial x}\right)(1+\ln z)}{(1+\ln z)^2}}$$

We need $\frac{\partial z}{\partial x}$ and $\frac{\partial y}{\partial x}$. keeping y ~~in terms of~~ const z

Differentiating (1) partially wrt x.

$$1 + \ln x + (1+\ln z)\frac{\partial z}{\partial x} = 0$$

$$\frac{\partial z}{\partial x} = \frac{-(1+\ln x)}{(1+\ln z)} \longrightarrow (4)$$

~~Diff (1) wrt partially wrt x keeping z const.~~

Rewriting (3)

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{-(1+\ln y)\frac{1}{z}\frac{\partial z}{\partial x}}{(1+\ln z)^2} \rightarrow (3)$$

$$\frac{\partial z}{\partial x} = \frac{-(1+\ln x)}{(1+\ln z)} \rightarrow (4)$$

Putting (4) in (3)

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{-(1+\ln y)\frac{1}{z}\frac{-(1+\ln x)}{(1+\ln z)}}{(1+\ln z)^2}$$

$$= \frac{1}{z}\frac{(-)(1+\ln y)(1+\ln x)}{(1+\ln z)^3}$$

at $x = y = z$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{-1}{x}\frac{(1+\ln x)^2}{(1+\ln x)^3} = \frac{-1}{x(1+\ln x)}$$

$$= \frac{-1}{x(\ln ex)} = -[x(\ln ex)]^{-1}$$

HW

6. If $u = (ar^n + br^{-n})(\cos n\theta + \sin n\theta)$, $\rightarrow (1)$
show that

$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0$$

Diff (1) partially w.r.t r.

$$u_r = \frac{\partial u}{\partial r} = (\cos n\theta + \sin n\theta)(anr^{n-1} - bnr^{-n-1})$$

$$u_r = (\cos n\theta + \sin n\theta)(anr^{n-1} - bnr^{-n-1})$$

$$(2) \leftarrow u_r = n(\cos n\theta + \sin n\theta)(ar^{n-1} - br^{-n-1})$$

Diff. (2) partially w.r.t r.

$$u_{rr} = n(\cos n\theta + \sin n\theta)(a(n-1)r^{n-2} + (b)(n+1)r^{-n-2})$$

$$(3) \leftarrow u_{rr} = n(\cos n\theta + \sin n\theta)(a(n-1)r^{n-2} + (b)(n+1)r^{-n-2})$$

Dividing (2) by r.

$$\frac{1}{r} u_r = \frac{n}{r}(\cos n\theta + \sin n\theta)(ar^{n-1} - br^{-n-1})$$

$$(4) \leftarrow \frac{1}{r} u_r = n(\cos n\theta + \sin n\theta)(ar^{n-2} - br^{-n-2})$$

Diff (1) partially w.r.t θ.

$$u_\theta = (ar^n + br^{-n})(-n \sin n\theta + n \cos n\theta)$$

$$u_\theta = n(ar^n + br^{-n})(\cos n\theta - \sin n\theta) \rightarrow (5)$$

Diff (5) partially w.r.t θ

$$u_{\theta\theta} = n(ar^n + br^{-n})(-n \sin n\theta - n \cos n\theta)$$

$$(6) \leftarrow u_{\theta\theta} = -n^2(ar^n + br^{-n})(\sin n\theta + \cos n\theta)$$

(1) $u = (ar^n + br^{-n})(\sin n\theta + \cos n\theta)$

(4) $\frac{1}{r} u_r = n(\cos n\theta + \sin n\theta)(ar^{n-2} - br^{-n-2})$

(3) $u_{rr} = n(\cos n\theta + \sin n\theta)(anr^{n-2} - ar^{n-2} + bnr^{-n-2} + br^{-n-2})$

$= n(\cos n\theta + \sin n\theta)(br^{-n-2} - ar^{n-2})$

$+ n^2(\cos n\theta + \sin n\theta)(ar^{n-2} + br^{-n-2})$

(6) $u_{\theta\theta} = -n^2(ar^n + br^{-n})(\sin n\theta + \cos n\theta)$

Dividing (6) by r^2

(7) $\frac{1}{r^2} u_{\theta\theta} = \frac{-n^2}{r^2}(ar^n + br^{-n})(\sin n\theta + \cos n\theta)$

$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta}$

$= \frac{n(\cos n\theta + \sin n\theta)}{r^2}(br^{-n} - ar^n)$

$+ \frac{n^2(\cos n\theta + \sin n\theta)}{r^2}(ar^n + br^{-n})$

$+ \frac{n(\cos n\theta + \sin n\theta)}{r^2}(ar^n - br^{-n})$

$$-\frac{n^2}{r^2}(ar^n + br^{-n})(\sin n\theta + \cos n\theta)$$

$= 0$

Hence proved

B. HOMOGENEOUS FUNCTIONS

- A real function $f(x,y)$ is called homogeneous of degree n if $f(\lambda x, \lambda y) = \lambda^n f(x,y)$ for all real λ.

- The constant n is called the degree of homogeneity.

- Alternatively, the function $f(x,y)$ is called homogeneous of degree n if $f(x,y) = x^n \phi\left(\frac{y}{x}\right)$ or $f(x,y) = y^n \phi\left(\frac{x}{y}\right)$

Eg: (1) $f(x,y) = \dfrac{x^5 - y^5}{x^3 + y^3}$

Then, $f(\lambda x, \lambda y) = \dfrac{\lambda^5 (x^5 - y^5)}{\lambda^3 (x^3 + y^3)} = \lambda^2 \dfrac{(x^5 - y^5)}{(x^3 + y^3)}$

$f(\lambda x, \lambda y) = \lambda^2 f(x,y)$

$\therefore$ f is homogeneous of degree 2.

OR

$$f(x,y) = \frac{x^5\left(1 - \left(\frac{y}{x}\right)^5\right)}{x^3\left(1 + \left(\frac{y}{x}\right)^3\right)}$$

$$= x^2 \left(\frac{1 - (y/x)^5}{1 + (y/x)^3}\right) = x^2 \phi\left(\frac{y}{x}\right)$$

$\therefore$ f is homogeneous of degree 2

(2) $f(x,y,z) = x^2y + y^2z + z^2x$

Then, $f(\lambda x, \lambda y, \lambda z) = \lambda^2 x^2 \lambda y + \lambda^2 y^2 \lambda z + \lambda^2 z^2 \lambda x$

$f(\lambda x, \lambda y, \lambda z) = \lambda^3 f(x,y,z)$

$\therefore$ f is homo. of degree 3.

OR

$$f(x,y,z) = x^3\left(\frac{y}{x}\right) + x^3\left(\frac{y^2}{x^2} \cdot \frac{z}{x}\right) + x^3\left(\frac{z^2}{x^2}\right)$$

$$= x^3\left(\frac{y}{x} + \left(\frac{y}{x}\right)^2 - \frac{z}{x} + \left(\frac{z}{x}\right)^2\right)$$

$$f(x,y,z) = x^3 \phi\left(\frac{y}{x}, \frac{z}{x}\right)$$

(3) $f(x,y) = \sin\left(\frac{x^3 - y^3}{x+y}\right)$

Then, $f(\lambda x, \lambda y) = \sin\left(\frac{\lambda^3 (x^3 - y^3)}{\lambda (x+y)}\right)$

$$= \sin\left(\lambda^2 \left(\frac{x^3 - y^3}{x+y}\right)\right) \neq \lambda^2 f(x,y)$$

$\therefore f(\lambda x, \lambda y) \neq \lambda^n f(x,y)$

$\therefore$ f is not homogeneous

find application

Euler's Theorem

- If u is a homogeneous function of x, y of degree n, then

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$$

Proof:

$$u = x^n \phi\left(\frac{y}{x}\right)$$

Diff partially wrt x:

$$\frac{\partial u}{\partial x} = nx^{n-1}\phi\left(\frac{y}{x}\right) + x^n \phi'\left(\frac{y}{x}\right)\left(\frac{-y}{x^2}\right)$$

$$x\frac{\partial u}{\partial x} = nx^n\phi\left(\frac{y}{x}\right) + x^n \phi'\left(\frac{y}{x}\right)\left(\frac{-y}{x}\right)$$

$$x\frac{\partial u}{\partial x} = nu + x^n \phi'\left(\frac{y}{x}\right)\left(\frac{-y}{x}\right) \longrightarrow (1)$$

Diff-partially wrt y:

$$\frac{\partial u}{\partial y} = x^n \phi'\left(\frac{y}{x}\right)\left(\frac{1}{x}\right)$$

$$y\frac{\partial u}{\partial y} = x^n \phi'\left(\frac{y}{x}\right)\left(\frac{y}{x}\right) \longrightarrow (2)$$

Adding (1) and (2)

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu + \cancel{x^n \phi'\left(\frac{y}{x}\right)\left(\frac{-y}{x}\right)} + \cancel{x^n \phi'\left(\frac{y}{x}\right)\left(\frac{y}{x}\right)}$$

$$\boxed{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu}$$

- Three variables:

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = nu$$

- In general, if u is a homogeneous function of $x_1, x_2, x_3, \ldots, x_k$ of degree n, then

$$\boxed{\sum_{i=1}^{k} x_i \frac{\partial u}{\partial x_i} = nu}$$

- He improved & used second partial derivatives

Extension of Euler's Theorem:

- If u is a homogeneous function of x & y of degree n, then

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u$$

u_{xx} $\qquad$ u_{yy}

$$\underbrace{xy\,u_{xy} + yx\,u_{yx}}_{\text{mixed derivatives are equal for homogeneous fn.}}$$

- coefficients are $(x+y)^2$, like $(x+y)^1$ in Euler's Theorem.
- Proof using Euler's Theorem

- For 3rd degree

co-efficients

x^3 . $3x^2y$ $3y^2x$ y^3

u_{xxy}
u_{xyx}
u_{yxx}

Geometry of homogeneous functions

for $z = f(x,y)$, homogeneous.

- Curves of cross-sections - level curves
- Slope of ~~function~~ tangent at (x,y) is the same as tangent at $(\lambda x, \lambda y)$

06-09-19

7. Verify Euler's theorem for $u = x^4y^2 \sin^{-1}\left(\frac{y}{x}\right)$

check homogenity

let $f(x,y) = u = x^4y^2 \sin^{-1}\left(\frac{y}{x}\right)$

$$f(\lambda x, \lambda y) = \lambda^4 x^4 \lambda^2 y^2 \sin^{-1}\left(\frac{\lambda y}{\lambda x}\right)$$

$$= \lambda^6 f(x,y)$$

$\therefore$ it is homogeneous of degree 6.

Acc to Euler's Theorem,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu = 6u$$

~~Di~~ $u = x^4 y^2 \sin^{-1}\left(\frac{y}{x}\right) \rightarrow (1)$

Diff. (1) partially w.r.t x

$$\frac{\partial u}{\partial x} = y^2\left[4x^3 \sin^{-1}\left(\frac{y}{x}\right) + \frac{x^4}{\sqrt{1-\frac{y^2}{x^2}}}\cdot\left(\frac{-1}{x^2}\right)(y)\right]$$

$$\frac{\partial u}{\partial x} = y^2\left[4x^3 \sin^{-1}\left(\frac{y}{x}\right) + \frac{x^{\cancel{4}\,3}\cdot \cancel{x}}{\sqrt{x^2-y^2}}\left(\frac{-y}{\cancel{x^2}}\right)\right] \rightarrow (a)$$

Multiplying (a) by x

$$x\frac{\partial u}{\partial x} = xy^2\left[4x^3 \sin^{-1}\left(\frac{y}{x}\right) \cancel{\theta} \frac{-x^3 y}{\sqrt{x^2-y^2}}\right]$$

$$x\frac{\partial u}{\partial x} = 4x^4y^2 \sin^{-1}\left(\frac{y}{x}\right) \frac{-x^4y^3}{\sqrt{x^2-y^2}} \quad \cancel{(2)}$$

$$\boxed{x\frac{\partial u}{\partial x} = 4u - \frac{x^4y^3}{\sqrt{x^2-y^2}}} \rightarrow (2)$$

Diff. (1) partially w.r.t y

$$\frac{\partial u}{\partial y} = x^4\left[2y \sin^{-1}\left(\frac{y}{x}\right) + \frac{y^2}{\sqrt{1-\frac{y^2}{x^2}}}\cdot\left(\frac{1}{x}\right)\right]$$

$$\frac{\partial u}{\partial y} = x^4\left[2y \sin^{-1}\left(\frac{y}{x}\right) + \frac{y^2}{\sqrt{x^2-y^2}}\right] \rightarrow (b)$$

Multiplying (b) by y

$$y\frac{\partial u}{\partial y} = 2x^4y^2\sin^{-1}\left(\frac{y}{x}\right) + \frac{x^4y^3}{\sqrt{x^2-y^2}}$$

~~Adding (1) and (2)~~

$\cancel{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}} =$

$$y\frac{\partial u}{\partial y} = 2u + \frac{x^4y^3}{\sqrt{x^2-y^2}} \longrightarrow (2)$$

Adding (1) and (2)

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 4u - \frac{x^4y^3}{\sqrt{x^2-y^2}} + 2u + \frac{x^4y^3}{\sqrt{x^2-y^2}}$$

$$\boxed{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 6u}$$

Hence proved.

8. If $u = e^{x/y}\cdot\sin\left(\frac{x}{y}\right) + e^{y/x}\cdot\cos\left(\frac{x}{y}\right)$.

~~(function is homo)~~

Find the value of $xu_x + yu_y$

Acc. to Euler's Theorem,

$$xu_x + yu_y = nu.$$

$$f(\lambda x, \lambda y) = f(x,y) = \lambda^0 f(x,y)$$

$$n = 0$$

$$\boxed{xu_x + yu_y = 0.}$$

9. If $u = \sin^{-1}\left(\frac{x^2y^2}{x+y}\right)$, prove that $xu_x + yu_y = 3\tan u$

Notice: u is not homo.

$$\sin u = \frac{x^2y^2}{x+y} = v. \longrightarrow (1)$$

$$v = \frac{x^2y^2}{x+y} = f(x,y)$$

$$f(\lambda x, \lambda y) = \frac{\lambda^2 x^2 \lambda^2 y^2}{\lambda(x+y)} = \lambda^3 f(x,y)$$

$\therefore n = 3.$ $v = \sin u$ is homo of deg 3.

$$\therefore x \cdot \frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = 3v$$

Replacing

$$x \cdot \left(\cos u \frac{\partial u}{\partial x}\right) + y\left(\cos u \frac{\partial u}{\partial y}\right) = 3\sin u$$

$$\boxed{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3\tan u}$$

Hence proved.

10. If $u = \cot^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, P.T. $xu_x + yu_y = \frac{-1}{4}\sin 2u$

$$\cot u = \frac{x+y}{\sqrt{x}+\sqrt{y}}$$

let $z = \cancel{\tan u = \frac{\sqrt{x}+\sqrt{y}}{x+y} = f(x,y)}$

$$\cancel{f(\lambda x, \lambda y) = \frac{\lambda^{1/2}}{\lambda}\frac{(\sqrt{x}+\sqrt{y})}{(x+y)}}$$

$\cancel{n = -1/2}$, ~~homo of deg -1/2~~

$$z = \cot u = f(x,y) = \frac{x+y}{\sqrt{x}+\sqrt{y}}$$

$$f(\lambda x, \lambda y) = \lambda^{1/2}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$$

homo of deg 1/2

$$\therefore x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial x} = \frac{1}{2}\cot u$$

$$x(-\operatorname{cosec}^2 u)u_x + y(-\operatorname{cosec}^2 u)u_y = \frac{1}{2}\cot u$$

$$xu_x + yu_y = \frac{-1}{2}\cot u \sin^2 u$$

$$\cancel{= \frac{-1}{2}\cos^2 u}$$

$$xu_x + yu_y = \frac{1}{2}\cos u \sin u = \frac{-1}{4}\sin 2u$$

$$\boxed{xu_x + yu_y = \frac{-1}{4}\sin 2u}$$

Hence proved

11. If $u = \dfrac{x^3y^3z^3}{x^3+y^3+z^3} + \ln\left(\dfrac{xy+yz+zx}{x^2+y^2+z^2}\right)$,

find the value of $xu_x + yu_y + zu_z$.

The function is not homogeneous.

12. let $V = \dfrac{x^3y^3z^3}{x^3+y^3+z^3}$ $\quad w = \ln\left(\dfrac{xy+yz+zx}{x^2+y^2+z^2}\right)$

homo. in deg 6 $\quad$ homo in deg. 0.

$n_1 = 6$ $\quad$ $n_2 = 0$.

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = x\left[\frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\right] + y\left[\frac{\partial w}{\partial y} + \frac{\partial v}{\partial y}\right]$$

$$= x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} + x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y}$$

$$= 6v + 0w = \boxed{\frac{6x^3y^3z^3}{x^3+y^3+z^3}} = xu_x + yu_y + zu_z$$

$f(x,y,z) = V = \dfrac{x^3y^3z^3}{x^3+y^3+z^3}$ $\quad f(\lambda x, \lambda y, \lambda z) = \dfrac{\lambda^9 x^3y^3z^3}{\lambda^3(x^3+y^3+z^3)}$

V is homo in deg 6.

$g(x,y,z) = w = \dfrac{xy+yz+zx}{x^2+y^2+z^2}$ $\quad g(\lambda x, \lambda y, \lambda z) = g(x,y,z)$

$36\left(\frac{1+4}{1+4}\right)$ $\frac{36\pi}{4}$

126 0 6 6

12. If $u = \frac{x^4+y^4}{x^2y^2} + x^6\tan^{-1}\left(\frac{x^2+y^2}{x^2+2xy}\right)$ find the value of

$$x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$$

at $x=1, y=2$.

$v = \frac{x^4+y^4}{x^2y^2}$ $\qquad w = x^6\tan^{-1}\left(\frac{x^2+y^2}{x^2+2xy}\right)$

$v_{\lambda x,\lambda y} = v_{x,y}$ $\qquad w_{\lambda x,\lambda y} = \lambda^6\, w_{x,y}$

homo in deg 0. $\qquad$ homo in deg. 6.

Extension theorem

$$x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$$

$$= x^2\frac{\partial^2 w}{\partial x^2} + y^2\frac{\partial^2 w}{\partial y^2} + 2xy\frac{\partial^2 w}{\partial x\partial y} + x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y}$$

$$= 6\times 5\, w + 6w = 36w$$

$$= 36x^6\tan^{-1}\left(\frac{x^2+y^2}{x^2+2xy}\right) = 36\tan^{-1}\left(\frac{1+4}{1+4}\right)$$

$$= 36\tan^{-1}(1) = \frac{36\pi}{4} = 9\pi.$$

T-S of $~~x(y)~~ = y + y^3 = x$

store
67

TOTAL DERIVATIVES

If $z = f(x,y)$ is a function of two independent variables x & y where $x = x(t)$, $y = y(t)$, then z is a function of a single varible t. The ordinary ~~der~~ derivative of z wrt t is called the <u>total derivative</u> of z wrt t and is given by

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

IMPLICIT FUNCTIONS

If $z = f(x,y)$ is an implicit relation between x & y, then the derivative of z wrt x is

$$\frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dx}$$

In particular, if $z = f(x,y) = c$, then its derivative is 0.

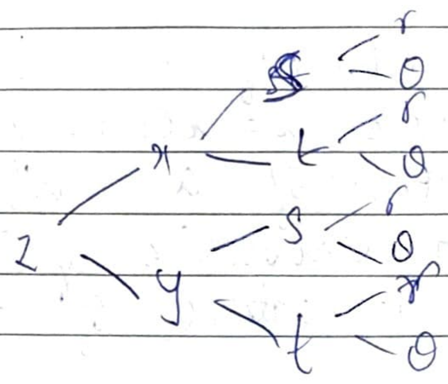
(2 function given; $z(x,y)$ and $f(y) = x$)

$$0 = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{-\partial z/\partial x}{\partial z/\partial y}$$

$$\frac{dy}{dx} = \frac{-\partial f/\partial x}{\partial f/\partial y}$$

COMPOSITE FUNCTIONS



If $z = f(x, y)$ where $x = x(s, t)$ and $y = y(s, t)$ then z is called a composite function

The derivatives of z wrt s and t are given by

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial s}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t}$$

no of terms = no. of ind. var. for z.
no. of eq. = no. of ind var for x, y - -

13. Find $\frac{dz}{dt}$ at $t = \pi/2$ if $z = e^{xy}$ where

$x = t\cos t \qquad y = t\sin t$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

$$\frac{dx}{dt} = \cos t - t\sin t.$$

$$\frac{dy}{dt} = \sin t + t\cos t.$$

at $t = \pi/2$

$x = \frac{\pi}{2}\cos\pi/2 = 0$

$x = 0$

$$\frac{\partial z}{\partial x} = y \cdot e^{xy}$$

$y = \frac{\pi}{2}\sin\pi/2$

$y = \pi/2$

$$\frac{\partial z}{\partial y} = x\, e^{xy}$$

$$\frac{dz}{dt} = (y e^{xy})(\cos t - t\sin t) + (x e^{xy})(\sin t + t\cos t)$$

$$= (t\sin t\, e^{t^2\cos t\sin t})(\cos t - t\sin t) + (t\cos t\, e^{t^2\sin t\cos t})(\sin t + t\cos t)$$

$$= \left(\frac{\pi}{2} e^{0}\right)\left(0 - \frac{\pi}{2}\right) + (0)$$

$$\boxed{\frac{dz}{dt} = -\frac{\pi^2}{4}}$$

14. If $z = xy^2 + x^2y$ where $x = at^2$, $y = 2at$, find $\frac{dz}{dt}$. Verify the result by direct substitution

using P.D.

$x = at^2$ $\qquad\qquad$ $y = 2at$.

Diff. wrt t $\qquad\qquad$ Diff. wrt t.

$\frac{dx}{dt} = 2at \rightarrow (1)$ $\qquad$ $\frac{dy}{dt} = 2a \rightarrow (2)$

$$z = xy^2 + x^2y \rightarrow (3)$$

Diff (3) partially wrt x.

$$\frac{\partial z}{\partial x} = y^2 + 2yx \rightarrow (4)$$

Diff. (3) partially wrt y

$$\frac{\partial z}{\partial y} = 2xy + x^2 \rightarrow (5)$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$

$$= (y^2+2yx)(2at) + (2xy+x^2)(2a)$$

$$= (4a^2t^2 + 4a^2t^3)(2at) + (4a^2t^3 + a^2t^4)(2a)$$

$$\frac{dz}{dt} = \underset{2at}{(4a^2t^2 + 4a^2t^3)} + 8a^3t^3 + 2a^3t^4$$

$$\frac{dz}{dt} = 8a^3t^3 + 8a^3t^4 + 8a^3t^3 + 2a^3t^4$$

$$\frac{dz}{dt} = 16a^3t^3 + 10a^3t^4 \rightarrow (a)$$

using direct substitution

$$z = (at^2)(2at)^2 + (at^2)^2(2at)$$
$$= (at^2)(4a^2t^2) + (a^2t^4)(2at)$$
$$= 4a^3t^4 + 2a^3t^5$$

$$\frac{dz}{dt} = 16a^3t^3 + 10a^3t^4 \longrightarrow (6)$$

(a) = (5). ⟹ verified result.

15. If x increases at the rate of $2\,cm\,s^{-1}$ at the instant when $x = 3\,cm$, $y = 1\,cm$, at what rate must y be changing in order that the function $2xy - 3x^2y$ shall be neither increasing nor decreasing?

let $z = 2xy - 3x^2y \longrightarrow (1)$

we know $\frac{dz}{dt} = 0$.

$\frac{dx}{dt} = 2cm^{-1}$ at $x = 3$, $y = 1\,cm$

$\frac{dy}{dt} = ?$

we know

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}.$$

Diff (1) partially wrt x,

$$\frac{\partial z}{\partial x} = 2y - 6yx. \longrightarrow (2)$$

Diff (1) partially wrt y.

$$\frac{\partial z}{\partial y} = 2x - 3x^2$$

$$\frac{\partial z}{\partial x} = 2y - 6yx \quad \text{at } (3,1)$$

$$= 2 - 18 \quad = -16$$

$$\boxed{\frac{\partial z}{\partial x} = -16}$$

$$\frac{\partial z}{\partial y} = 2x - 3x^2 \quad \text{at } (3,1)$$

$$= 6 - 27$$

$$\boxed{\frac{\partial z}{\partial y} = -21}$$

$$0 = (-16)(2) - (21)\left(\frac{dy}{dt}\right)$$

$$\frac{-16 \times 2}{21} = \boxed{\frac{dy}{dt} = \frac{-32}{21}}$$

y decreases at a rate of $-32/21$ cm s^{-1}

16. If $z = 2xy^2 - 3x^2y$ and if x increases at the rate of 2 cm s^{-1} and it passes through $x = 3$ cm, show that if y is passing through the value of $y = 1$ cm, then y must be decreasing at the rate of $2\,^2/_{15}$ cm s^{-1} in order that z shall remain constant.

$$z = 2xy^2 - 3x^2y = k$$

We know $\frac{dz}{dt} = 0.$

$\frac{dx}{dt} = 2 \text{ cm}^{-1}$ $\frac{dy}{dt} = ?$

$(3, 1)$

$$\frac{\partial z}{\partial x} = 2y^2 - 6xy = 2 - 6\times 3 = -16$$

$$\frac{\partial z}{\partial y} = 4xy - 3x^2 = 4\times 3 - 27 = 12 - 27 = -15$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}.$$

$$0 = (-16)(2) + (-15)\left(\frac{dy}{dt}\right).$$

$$\boxed{\frac{-32}{15} = \frac{dy}{dt} = 2\tfrac{2}{15}.}$$

Hence verified

17. If $u = x\log_e(xy)$ where $x^3 + y^3 - 3axy = 1$, find $\frac{du}{dx}$

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dx}.$$

$$x^3 + y^3 - 3axy = 1 \longrightarrow (1)$$
$$u = x\ln(xy) \longrightarrow (2)$$

Diff (1) wrt x.

$$3x^2 + 3y^2\frac{dy}{dx} - 3a\left(y + x\frac{dy}{dx}\right) = 0$$

$$x^2 + y^2\frac{dy}{dx} = a\left(y + x\frac{dy}{dx}\right)$$

$$\frac{dy}{dx}\left(y^2 - ax\right) = ay - x^2$$

$$\boxed{\frac{dy}{dx} = \frac{ay - x^2}{y^2 - ax}} \longrightarrow (3)$$

$u = x\ln(xy)$

Diff. (2) partially wrt x

$$\frac{\partial u}{\partial x} = \ln(xy) + \frac{x}{xy}(y)$$

$$\boxed{\frac{\partial u}{\partial x} = \ln(xy) + 1} \longrightarrow (4)$$

Diff (2) partially wrt y

$$\frac{\partial u}{\partial y} = \frac{x \cdot x}{xy} = \frac{x}{y}$$

$$\boxed{\frac{\partial u}{\partial y} = \frac{x}{y}} \longrightarrow (5)$$

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dx}$$

$$\frac{du}{dx} = \ln(xy) + 1 + \left(\frac{x}{y}\right)\left(\frac{ay - x^2}{y^2 - ax}\right)$$

$$\boxed{\frac{du}{dx} = 1 + \ln(xy) + \frac{x}{y}\left(\frac{ay - x^2}{y^2 - ax}\right)}$$

18. If $y \ln(\cos x) = x \ln(\sin y)$, find $\frac{dy}{dx}$.

~~Diff bs wrt. x.~~

Let $u = y \ln(\cos x) - x \ln(\sin y) = 0 \rightarrow (1)$

u is a constant:

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx}$$

$$0 = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{-\partial u/\partial x}{\partial u/\partial y}$$

Diff (1) ~~wrt~~ partially wrt x.

$$\frac{\partial u}{\partial x} = \frac{y(-\sin x)}{\cos x} - \ln(\sin y)$$

$$\frac{\partial u}{\partial x} = (-\tan x)y - \ln(\sin y) \longrightarrow (2)$$

Diff (1) partially wrt y

$$\frac{\partial u}{\partial y} = \ln(\cos x) \frac{-x \cos y}{\sin y}$$

$$\frac{\partial u}{\partial y} = \ln(\cos x) - x \cot y \longrightarrow (3)$$

$$\therefore \frac{dy}{dx} = \frac{-\partial u/\partial x}{\partial u/\partial y}$$

$$= \frac{+((\tan x) y + \ln(\sin y))}{\ln(\cos x) - x \cot y}$$

$$\boxed{\frac{dy}{dx} = \frac{y\tan x + \ln(\sin y)}{\ln(\cos x) - x\cot y}}$$

19. Find the total differential coefficient of x^2y with respect to x when x and y are connected by the relation

$$x^2 + xy + y^2 = 1 \longrightarrow (1)$$

Let $z = x^2y \longrightarrow (2)$

$$\frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dx}$$

Diff. (1) wrt x.

$$2x + y + x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(x+2y) = -2x - y$$

$$\boxed{\frac{dy}{dx} = \frac{-2x-y}{x+2y}}$$

Diff (2) partially wrt x.

$$\frac{\partial z}{\partial x} = 2xy$$

Diff. (2) partially wrt y

$$\frac{\partial z}{\partial y} = x^2.$$

$$\boxed{\frac{dz}{dx} = 2xy - x^2\left(\frac{2x+y}{x+2y}\right)}$$

20. If $u = f\left(\frac{y-x}{yx}, \frac{z-x}{zx}\right)$, show that

$$x^2 u_x + y^2 u_y + z^2 u_z = 0.$$

Let $s = \frac{y-x}{yx} \to (1)$, $t = \frac{z-x}{zx} \to (2)$.

$$u = f(s, t) \to (3)$$

$$\cancel{\frac{\partial u}{\partial x} = u_x = u_s \cdot \frac{ds}{dx} + u_t \frac{dt}{dx}}.$$

$$\cancel{\frac{\partial u}{\partial x} = \frac{\partial u}{\partial s} \cdot \frac{ds}{dx} + \frac{\partial u}{\partial t} \frac{dt}{dx}}.$$

~~Diff (1) wrt x.~~

$$\cancel{s = \frac{y-x}{yx}}$$

$$\cancel{\frac{ds}{dx} = \frac{\left(\frac{dy}{dx} - 1\right)(yx) - (y-x)\left(\frac{dy}{dx} \cdot x +\right.}{(yx)^2}}$$

$$\cancel{\frac{ds}{dx}} =$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial s} \cdot \frac{\partial s}{\partial x} + \frac{\partial u}{\partial t} \cdot \frac{\partial t}{\partial x}$$

$$s = \frac{y-x}{yx} = \frac{1}{x} - \frac{1}{y} \rightarrow (4)$$

Diff (4) wrt x partially

$$\frac{\partial s}{\partial x} = \frac{-1}{x^2} \rightarrow (a)$$

~~Diff (4) partially wrt y~~

$$\cancel{\frac{\partial s}{\partial y} = \frac{+1}{y^2} \rightarrow (b)}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial s} \cdot \frac{\partial s}{\partial x} + \frac{\partial u}{\partial t} \cdot \frac{\partial t}{\partial x}$$

$$t = \frac{z-x}{zx} = \frac{1}{x} - \frac{1}{z}$$

$$\frac{\partial t}{\partial x} = \frac{-1}{x^2} \qquad \frac{\partial t}{\partial z} = \frac{1}{z^2}$$

$$\therefore u_x = u_s \cdot \left(\frac{-1}{x^2}\right) + u_t\left(\frac{-1}{x^2}\right)$$

$$u_x = \frac{-1}{x^2}(u_s + u_t)$$

$$\boxed{x^2 u_x = -u_s - u_t} \longrightarrow (a)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial y}$$

$$\frac{\partial s}{\partial y} = \frac{1}{y^2} \qquad \frac{\partial t}{\partial y} = 0$$

$$\therefore \frac{\partial u}{\partial y} = \frac{\partial u}{\partial s}\left(\frac{1}{y^2}\right)$$

$$\boxed{y^2 u_y = u_s} \longrightarrow (b)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial z} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial z}$$

$$u_z = u_t\left(\frac{1}{z^2}\right) \Rightarrow \boxed{z^2 u_z = u_t} \longrightarrow (c)$$

Adding (a), (b), & (c)

$$\boxed{x^2 u_x + y^2 u_y + z^2 u_z = 0}$$

Q1. If $z = f(x, y)$ where $x = e^u \cos v$, $y = e^u \sin v$
show that

(a) $y \frac{\partial z}{\partial u} + x \frac{\partial z}{\partial v} = e^{2u} \left(\frac{\partial z}{\partial y} \right)$

(b) $\left(\frac{\partial z}{\partial u} \right)^2 + \left(\frac{\partial z}{\partial v} \right)^2 = e^{2u} \left(\left(\frac{\partial z}{\partial x} \right)^2 + \left(\frac{\partial z}{\partial y} \right)^2 \right)$

~~Diff. given equat~~

$x = e^u \cos v$ $\qquad$ $y = e^u \sin v$

$\frac{\partial x}{\partial u} = e^u \cos v$ $\qquad$ $\frac{\partial y}{\partial u} = e^u \sin v$

$\frac{\partial x}{\partial v} = -e^u \sin v$ $\qquad$ $\frac{\partial y}{\partial v} = e^u \cos v$

Using chain rule, w.r.t u

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u}$$

$$= \frac{\partial z}{\partial x} e^u \cos v + \frac{\partial z}{\partial y} e^u \sin v$$

$$y \frac{\partial z}{\partial u} = y e^u \left(\frac{\partial z}{\partial x} \cos v + \frac{\partial z}{\partial y} \sin v \right) \longrightarrow (1)$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\left(-e^u \sin v\right) + \frac{\partial z}{\partial y}\left(e^u \cos v\right)$$

$$x\frac{\partial z}{\partial v} = xe^u\left(-\frac{\partial z}{\partial x}\sin v + \frac{\partial z}{\partial y}\cos v\right) \rightarrow (2).$$

(a)
Adding (1) & (2):

$$y\frac{\partial z}{\partial u} + x\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\left(ye^u\cos v - xe^u\sin v\right) + \frac{\partial z}{\partial y}\left(xe^u\cos v + ye^u\sin v\right)$$

But $x = e^u\cos v$, $y = e^u\sin v$

$$y\frac{\partial z}{\partial u} + x\frac{\partial z}{\partial v} = \cancel{\frac{\partial z}{\partial x}\left(yx - xy\right)}^{\,0} + \frac{\partial z}{\partial y}\left(x^2+y^2\right)$$

$$y\frac{\partial z}{\partial u} + x\frac{\partial z}{\partial v} = \frac{\partial z}{\partial y}\left(e^{2u}\cos^2 v + e^{2u}\sin^2 v\right)$$

$$\boxed{y\frac{\partial z}{\partial u} + x\frac{\partial z}{\partial v} = \frac{\partial z}{\partial y}\left(e^{2u}\right)}$$

Hence proved.

(b) $\frac{\partial z}{\partial u} = xy\frac{\partial z}{\partial x} + y^2\frac{\partial z}{\partial y} \rightarrow (3)$

$\frac{\partial z}{\partial v} = -xy\frac{\partial z}{\partial x} + x^2\frac{\partial z}{\partial y} \rightarrow (4)$

squaring (3)

$$z_u^2 = \left(\frac{\partial z}{\partial u}\right)^2 = x^2y^2z_x^2 + y^4z_y^2 + 2xy^3z_xz_y$$

squaring (4)

$$z_v^2 = x^2y^2z_x^2 + x^4z_y^2 - 2x^3yz_xz_y$$

Adding them,

$$z_u^2 + z_v^2 = 2x^2y^2z_x^2 + z_y^2(x^4+y^4) + 2xyz_xz_y(y^2-x^2)$$

$\downarrow$

equation (5)

$$\cancel{(z_x^2+z_y^2)\, e^{2u} = (z_x^2+z_y^2)(x^2+y^2)}$$

(b) $\dfrac{\partial z}{\partial u} = e^u \cos v \dfrac{\partial z}{\partial x} + e^u \dfrac{\partial z}{\partial y} \sin v$

$$\frac{\partial z}{\partial u} = x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} \longrightarrow 3$$

$$\frac{\partial z}{\partial v} = -e^u \sin v \frac{\partial z}{\partial x} + e^u \cos v \frac{\partial z}{\partial y}$$

$$= -y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y} \longrightarrow 4$$

squaring 3 & 4.

$$z_u^2 + z_v^2 = x^2 z_x^2 + y^2 z_y^2 + \cancel{2xy z_x z_y}$$

$$+ y^2 z_x^2 + x^2 z_y^2 \cancel{-2xy z_x z_y}$$

$$= z_x^2(x^2+y^2) + z_y^2(x^2+y^2)$$

$$z_u^2 + z_v^2 = (x^2+y^2)(z_x^2+z_y^2)$$

$$= (z_x^2+z_y^2)(e^{2u}\cos^2 v + e^{2u}\sin^2 v)$$

$$z_u^2 + z_v^2 = (z_x^2+z_y^2)(e^{2u})$$

$$\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2 = e^{2u}\left(\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right)$$

22 If $u = f(r, s, t)$ where $r = \frac{x}{y}$, $s = \frac{y}{z}$ and $t = \frac{z}{x}$, ~~show that~~. find the value of $xu_x + yu_y + zu_z$.

we know:

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial x}$$

$$\frac{\partial r}{\partial x} = \frac{1}{y};\ \frac{\partial r}{\partial y} = \frac{-x}{y^2};\ \frac{\partial r}{\partial z} = 0.$$

$$\frac{\partial s}{\partial x} = 0;\ \frac{\partial s}{\partial y} = \frac{1}{z};\ \frac{\partial s}{\partial z} = \frac{-y}{z^2}$$

$$\frac{\partial t}{\partial x} = \frac{-z}{x^2}\ \ \frac{\partial t}{\partial y} = 0;\ \frac{\partial t}{\partial z} = \frac{1}{x}$$

$$u_x = \frac{u_r}{y} + u_s(0) + u_{t}\left(\frac{-z}{x^2}\right)$$

$$xu_x = \frac{x}{y}u_r \cancel{+0} - u_t\frac{z}{x}$$

$$xu_x = ru_r - tu_t \quad \text{--- (1)}$$

$$u_y = u_r r_y + u_s s_y + u_t t_y$$

$$= u_r\left(\frac{-x}{y^2}\right) + u_s\left(\frac{1}{z}\right) + 0.$$

$$y u_y = -u_r r + s u_s \longrightarrow (2)$$

$$u_z = u_r r_z + u_s s_z + u_t t_z$$

$$= u_r 0 + u_s\left(\frac{-y}{z^2}\right) + u_t\left(\frac{1}{x}\right)$$

$$z u_z = -u_s(s) + t u_t \longrightarrow (3)$$

Adding (1), (2), (3)

$$\boxed{x u_x + y u_y + z u_z = 0}$$

11.09.19

23. By changing the independent variables u and v to x and y by means of the relations
$x = u\cos\alpha - v\sin\alpha$
$y = u\sin\alpha + v\cos\alpha$
show that $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}$ transforms to $\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}$

~~Let z(x,y) be a function of x & y.~~

$$x = u\cos\alpha - v\sin\alpha \longrightarrow (1)$$
$$y = u\sin\alpha + v\cos\alpha \longrightarrow (2)$$

We need u & v in terms of x & y.

~~$x\cos\alpha =$~~

(1) $\times \cos\alpha +$ (2) $\sin\alpha$

$$x\cos\alpha + y\sin\alpha = u \longrightarrow (3)$$

(1) $\times -\sin\alpha +$ (2) $\times \cos\alpha$

$$-x\sin\alpha + y\cos\alpha = v \longrightarrow (4)$$

$$u = x\cos\alpha + y\sin\alpha$$
$$v = y\cos\alpha - x\sin\alpha$$

$$\begin{tikzcd}
 & & x \\
 & u \arrow[ur] \arrow[r] & y \\
z \arrow[ur] \arrow[dr] & & x \\
 & v \arrow[ur] \arrow[r] & y
\end{tikzcd}$$

let $z(u,v)$ where $u(x,y)$ & $v(x,y)$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}(\cos\alpha) + \frac{\partial z}{\partial v}(-\sin\alpha)$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\cos\alpha - \frac{\partial z}{\partial v}\sin\alpha\right)$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\cos\alpha - \frac{\partial z}{\partial v}\sin\alpha\right)\cdot\frac{\partial u}{\partial x}$$
$$+ \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\cos\alpha - \frac{\partial z}{\partial v}\sin\alpha\right)\frac{\partial v}{\partial x}.$$

$$= \left[\frac{\partial^2 z}{\partial u^2}\cos\alpha - \frac{\partial^2 z}{\partial u\partial v}\sin\alpha\right]\frac{\partial u}{\partial x}$$
$$+ \left[\frac{\partial^2 z}{\partial v\partial u}\cos\alpha - \frac{\partial^2 z}{\partial v^2}\sin\alpha\right]\frac{\partial v}{\partial x}$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial u^2}\cos^2\alpha - \frac{\partial^2 z}{\partial u\partial v}\sin\alpha\cos\alpha$$
$$- \frac{\partial^2 z}{\partial v\partial u}\cos\alpha\sin\alpha + \frac{\partial^2 z}{\partial v^2}\sin^2\alpha$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\sin\alpha + \frac{\partial z}{\partial v}.\cos\alpha$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\sin\alpha + \frac{\partial z}{\partial v}\cos\alpha\right)$$

$$= \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\sin\alpha + \frac{\partial z}{\partial v}\cos\alpha\right)\frac{\partial u}{\partial y}$$

$$+ \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\sin\alpha + \frac{\partial z}{\partial v}\cos\alpha\right)\frac{\partial v}{\partial y}.$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 z}{\partial u^2}\sin^2\alpha + \frac{\partial z}{\partial u\partial v}\sin\alpha\cos\alpha. + \frac{\partial^2 z}{\partial v^2}\cos^2\alpha$$
$$+ \frac{\partial^2 z}{\partial v\partial u}\sin\alpha\cos\alpha$$

$$\boxed{\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}}$$

MAXIMA & MINIMA OF MULTIVARIABLE FUNCTIONS

$z = f(x, y)$ is a surface in 3 dimension.

Maxima & minima of functions of two variables.

A function $f(x, y)$ of two independent variables x and y is said to have a maximum (or minimum) at (a, b) if

$$f(a, b) > f(a+h, b+k)$$

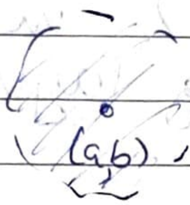
$$\left(\text{or} \quad f(a, b) < f(a+h, b+k) \right)$$

for all positive and negative small values of h & k.

For one variable, $y = f(x)$; $\frac{dy}{dx} = 0$,

$\frac{d^2y}{dx^2}$ at points

(infinitely many points)



consider a set of points surrounding (a,b) as $(a+h, b+k) \leftarrow$ all pts in vicinity

For x : $(x-h, x+h)$, for y : $(y-k, y+k)$

Taylor's Theorem for Two Variables

Recall, $f(a+h) = f(a) + \frac{h}{1!} f'(a) + \frac{h^2}{2!} f''(a) + \dots$

$$+ \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a)$$

$$+ \frac{h^n}{n!} f^{(n)}(a+\theta h)$$

Let f be a function of x, y

$f(x,y)$ ~~can be a~~ can be approximated as.

$\cancel{f(x+a, y+b) = f(x,y) + ((x+a)f_x + (y+b)}$

$$f(x,y) = f(a,b) + \frac{1}{1!}\left((x-a)f_x(a,b) + (y-b)f_y(a,b)\right)$$
$$+ \frac{1}{2!}\left((x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\right) + \dots$$

Q: Expand $x^2y+3y-2$ in powers of $(x-1)$ and $(y+2)$ using Taylor's Theorem.

$f(x,y) = x^2y+3y-2$

$f(1,-2) = -2-6-2 = \boxed{-10}$

at $(1,-2)$

$f_x(x,y) = 2xy$ $\quad -4$

$f_y(x,y) = x^2+3$ $\quad 4$

$f_{xx}(x,y) = 2y$ $\quad -4$

$f_{yy}(x,y) = 0$ $\quad 0$

$f_{yx}(x,y) = 2x$ $\quad 2$

$f_{xxx} = 0$ $\quad 0$

$f_{yyy} = 0$ $\quad 0$

$f_{yxx} = 2$ $\quad 2$

$f_{yyx} = 0$ $\quad 0$

$$f(x,y) = -10 + \frac{1}{1!}\left((x-1)(-4) + 4(y+2)\right)$$

$$+\frac{1}{2!}\left(-4(x-1)^2 + 4(x-1)(y+2)\right) + \frac{1}{3!}\left(6(x-1)^2(y+2)\right)$$

Working Rule to find the extremum of $f(x, y)$

(1) Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$. Solve the equations $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$ and find the solutions (critical points) (a_1, b_1), (a_2, b_2), (a_3, b_3) ...

(2) Find $r = \frac{\partial^2 f}{\partial x^2}$, $s = \frac{\partial^2 f}{\partial x \partial y}$, $t = \frac{\partial^2 f}{\partial y^2}$

for continuous, mixed are same.

and evaluate them at the critical points.

(3) If $rt - s^2$ (from Taylor's Theorem) > 0 and $r > 0$ at (a, b), then the function f has a minimum at (a, b)

(4) If $rt - s^2 > 0$, and $r < 0$ at (a, b), then the function f has a maximum at (a, b).

(5) If $rt - s^2 < 0$ at (a, b), then the function f has neither a maximum, nor a minimum at (a, b). Such critical points are called saddle points.

(6) If $rt - s^2 = 0$, then nothing can be said about the maximum or min. at (a, b)

Saddle points

24. Find the extreme values of $x^3y^2(12-3x-4y)$

Let $f(x,y) = x^3y^2(12-3x-4y)$

$$f(x,y) = 12x^3y^2 - 3x^4y^2 - 4x^3y^3$$

$$\therefore \frac{\partial f}{\partial x} = 36x^2y^2 - 12x^3y^2 - 12x^2y^3$$

$$\frac{\partial f}{\partial x} = 12x^2y^2(3-x-y)$$

$$\therefore \frac{\partial f}{\partial y} = 24x^3y - 6x^4y - 12x^3y^2$$

$$\frac{\partial f}{\partial y} = 6x^3y(4-x-2y)$$

At the critical point, $\partial f/\partial x = 0$, $\partial f/\partial y = 0$

$$12x^2y^2(3-x-y) = 0$$

$x = 0$, $y = (-\infty, \infty)$

$y = 0$, $x = (-\infty, \infty)$

Why are mixed = for cont. fn.

$$x+y=3 \Rightarrow y=3-x$$
$$x+2y=4$$

$$x+6-2x=4$$
$$6-x=4$$
$$\boxed{x=2,\ y=1}$$

∴ the only stationary point is (2, 1)

$$r=\frac{\partial^2 f}{\partial x^2} \qquad s=\frac{\partial^2 f}{\partial x\partial y} \qquad t=\frac{\partial^2 f}{\partial y^2}$$

$$r=\frac{\partial^2 f}{\partial x^2}=72xy^2-36x^2y^2-24xy^3$$

$$s=\frac{\partial^2 f}{\partial x\partial y}=72x^2y-24x^3y-36x^2y^2$$

$$t=\frac{\partial^2 f}{\partial y^2}=24x^3-6x^4-24x^3y$$

At (2, 1), $r=(72)(2)(1)-36(4)(1)-24(2)(1)$
$$r=144-144-48$$

$$r=-48$$

$$s=(72)(4)-(24)(8)-(36)(4)$$
$$=(72)(2)-(24)(8)$$
$$=(24)[6-8]=-48.$$

$t = \cancel{(24)(8)} - (6)(16) - \cancel{(24)(8)}$

$t = -48 - 48$

$t = -96$

$r = -48, \quad s = -48, \quad t = -96.$

$rt - s^2 = (-48)(-96) - (48)^2$

$= 2304$

$rt - s^2 > 0, \quad r < 0.$

the function f has a max.

at $(2,1)$

$f(x,y) = x^3y^2 (12 - 3x - 4y)$

$f(2,1) = (8)(1)(12 - 6 - 4) = 16.$

max value $= 16$

Plot in maxima

25. Find the points on the surface $z^2 = xy + 1$. nearest to the origin. Also find that distance.

Let a point be (x,y,z)

(x,y,z) to $(0,0,0)$

Any point on the surface $z^2 = xy + 1$ is (x,y,z)

$\cancel{x^2 + y^2 + z^2}$

distance $R = \sqrt{x^2 + y^2 + z^2}$

$R^2 = \sqrt{x^2 + y^2 + z^2}$ ⟸ minimum.

or $x^2 + y^2 + z^2$ is ~~f~~ min.

but $z^2 = xy + 1$

$x^2 + y^2 + xy + 1$

Let $f(x, y) = x^2 + y^2 + xy + 1$

$\frac{\partial f}{\partial x} = 2x + y$

$\frac{\partial f}{\partial y} = 2y + x$

At extremetiles, $\partial f/\partial x = 0$, $\partial f/\partial y = 0$

$2x + y = 0$ $\quad 2y + x = 0 \Rightarrow x = -2y$

$2(-2y) + y = 0$

$-4y + y = 0 \Rightarrow -3y = 0 \Rightarrow y = 0$

$\therefore x = 0$

$\therefore$ the critical point is $(0, 0)$

$r = \frac{\partial^2 f}{\partial x^2} = 2 \qquad t = \frac{\partial^2 f}{\partial y^2} = 2$

$s = \frac{\partial^2 f}{\partial x \partial y} = 1$

$rt - s^2 = (2)(2) - 1 = 3 > 0$

$rt - s^2 > 0$ and $r > 0 \Rightarrow f$ is min. at $(x, y) = (0,0)$

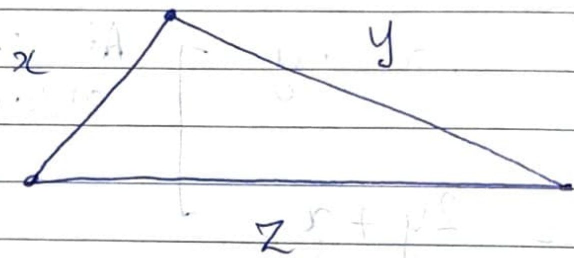
$\therefore$ on the surface, the points $(0, 0, z)$ are minima. $z^2 = 1 \Rightarrow z = \pm 1$.

$(0, 0, -1)$ and $(0, 0, 1)$.

$R = \sqrt{x^2 + y^2 + xy + 1} = \boxed{1 = R}$

186

26. Prove that if the perimeter of a triangle is constant, then its area is the maximum when the triangle is equilateral



Let the three sides be x, y, z.
Let the perimeter $x+y+z = 2k$.
$z = 2k - x - y$

Area of the Δ is

$$A = \sqrt{k(k-x)(k-y)(k-z)}$$

$$A^2 = k(k-x)(k-y)(k-z)$$
$$= k(k-x)(k-y)(k-2k+x+y)$$
$$= k(k-x)(k-y)(x+y-k)$$
$$= \cancel{(k^2-kx)(k-y)(x+y-k)}$$
$$= (k^3 - k^2x - k^2y + kxy)(x+y-k)$$

$$A^2 = (k^3x - k^2x^2 - k^2xy + kx^2y + k^3y - k^2xy - k^2y^2 + kxy^2 - k^4 + k^3x + k^3y - k^2xy)$$

The expression for A^2 should be minimised.

Let $f(x,y) = k^3x - k^2x^2 - k^2xy + kx^2y + k^3y - k^2xy - k^2y^2 + kxy^2 - k^4 + k^3x + k^3y - k^2xy$

$$\frac{\partial f}{\partial x} = k^3 - \cancel{2k^2x - k^2y} + \cancel{2kxy} - k^2y + ky^2 + k^3 - k^2y.$$

$$= \cancel{2k^3} - 2k^2x - \cancel{3k^2y} + 2kxy + ky^2 = 0$$

$$= 2k^2 - 2kx - 3ky + 2xy + y^2 = 0$$

$$f(x,y) = k(k-x)(k-y)(x+y-k)$$

$$\frac{\partial f}{\partial x} = k(k-y)\left[(-1)(x+y-k) + (k-x)(1)\right]$$

$$= k(k-y)(k-x-y+k-x)$$

$$\frac{\partial f}{\partial x} = k(k-y)(2k-2x-y) \longrightarrow 1$$

$$\frac{\partial f}{\partial y} = k(k-x)\left[(-1)(x+y-k) + (k-y)(1)\right]$$

$$= k(k-x)\{k-x-y+k-y\}$$

$$\frac{\partial f}{\partial y} = k(k-x)(2k-2y-x) \longrightarrow 2$$

$$2k - 2x - y = 0$$

$$2k - 2y - x = 0.$$

$$y = 2k - 2x.$$

$$2k - 2(2k-2x) - x = 0$$

$$2k - 4k + 4x - x = 0$$

$$-2k + 3x = 0$$

$$\boxed{x = \frac{2k}{3}}$$

$$y = 2k - \frac{4k}{3}$$

$$\boxed{y = \frac{2k}{3}}$$

$$r = \frac{\partial^2 f}{\partial x^2} = k(k-y)(-2)$$

$$s = \frac{\partial^2 f}{\partial x \partial y} = k\left[(-1)(2k-2y-x) + (k-x)(-1)\right]$$

$$s = k(\cancel{x} + 2y - 2k - k + \cancel{x})$$

$$s = k(\cancel{2x}\ 2y - k)$$

$$t = \frac{\partial^2 f}{\partial y^2} = k(k-x)(-2)$$

At $\left(\frac{2k}{3}, \frac{2k}{3}\right)$

$$r = k\left(\frac{k}{3}\right)(-2) = \boxed{\frac{-2k^2}{3} = r}$$

$$s = k\left(\frac{4k}{3} - k\right) = \boxed{\frac{k^2}{3} = s}$$

$$t = k\left(\frac{k}{3}\right)(-2) = \boxed{\frac{2k^2}{3} = t}$$

$r < 0$

$$rt - s^2 = \frac{4k^4}{9} - \frac{k^4}{9} = \frac{k^4}{3} > 0$$

$\therefore$ max. at $(2k/3, 2k/3)$

$$z = 2k - x - y = 2k - \frac{2k}{3} - \frac{2k}{3} = \frac{2k}{3}$$

$$x = \frac{2k}{3}, \quad y = \frac{2k}{3}, \quad z = \frac{2k}{3}$$

$\therefore$ the Δ is equilatural when it has max. area.

Limitations

1. Only for 2 variables

LAGRANGE'S UNDETERMINED MULTIPLIERS METHOD

Let $u = f(x, y, z)$ be a function of x, y, z which are connected by the relation $\phi(x, y, z) = c$.

To find the extremum of $f(x, y, z)$ subject to the condition $\phi = c$.

At the critical point,

$$\frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = 0, \quad \frac{\partial f}{\partial z} = 0$$

$$\Rightarrow \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} \cdot dy + \frac{\partial f}{\partial z} \cdot dz = 0 \quad \text{—(1)}$$

$$\boxed{df = 0}$$

exact differential / ~~total~~ differential

(differential is not wrt var; only derivative is) consider $\phi = c$

$$d\phi = 0.$$

$$\frac{\partial \phi}{\partial x} \cdot dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz = 0 \rightarrow (2)$$

$(1) + \lambda(2)$ ← can also be a function of (x, y, z)

$$\left(\frac{\partial f}{\partial x} + \lambda \frac{\partial \phi}{\partial x}\right) dx + \left(\frac{\partial f}{\partial y} + \lambda \frac{\partial \phi}{\partial y}\right) dy + \left(\frac{\partial f}{\partial z} + \lambda \frac{\partial \phi}{\partial z}\right) dz = 0$$

(where λ is called the Lagrange's Multiplier)

This equation is satisfied only when

$$\frac{\partial f}{\partial x} + \lambda \frac{\partial \phi}{\partial x} = 0, \quad \frac{\partial f}{\partial y} + \lambda \frac{\partial \phi}{\partial y} = 0, \quad \frac{\partial f}{\partial z} + \lambda \frac{\partial \phi}{\partial z} = 0.$$

as dx, dy & dz are independent of each other

We need to find x, y, z using the 3 equations and $\phi = c$.

conclusion,

$$\frac{\partial f}{\partial x} + \lambda \frac{\partial \phi}{\partial x} = 0$$

$$\frac{\partial f}{\partial y} + \lambda \frac{\partial \phi}{\partial y} = 0$$

$$\frac{\partial f}{\partial z} + \lambda \frac{\partial \phi}{\partial z} = 0$$

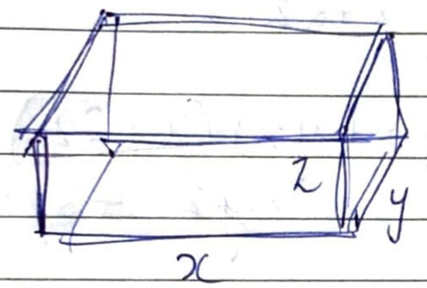
These eqn. can be solved simultaneously along with $\phi = c$ if necessary to get the critical points x, y, z

Drawbacks:

Cannot get to know whether it is max/min

27. Find the dimensions of a rectangular box of maximum capacity whose surface area is given ~~and~~ when
(a) the box is open at the top
(b) the box is closed on all sides

(a)



$A = xy + 2(yz) + 2(zx)$

$A = xy + 2yz + 2zx$

Volume $\boxed{V = xyz}$

to be extremised.

~~Let $f(x,y,z) = xy + 2yz + 2zx$~~

~~$\phi(x,y,z) = xy + 2yz + 2zx = A$~~

Let $f(x,y,z) = xyz = V$

$\phi(x,y,z) = xy + 2yz + 2zx. = A.$

To find the max. of $f(x,y,z)$ subject to the condition $\phi(x,y,z) = A$.

Let $F = f + \lambda\phi$

$= xyz + \lambda(xy + 2yz + 2zx)$

At a critical point,

$\frac{\partial F}{\partial x} = 0, \quad \frac{\partial F}{\partial y} = 0, \quad \frac{\partial F}{\partial z} = 0$

$\Rightarrow \frac{\partial F}{\partial x} = yz + \lambda(y + 2z) = 0$

$\frac{\partial F}{\partial y} = xz + \lambda(x + 2z) = 0$

$\frac{\partial F}{\partial z} = xy + \lambda(2y + 2x) = 0.$

$$\Rightarrow \frac{-yz}{y+2z} = \frac{-xz}{x+2z} = \frac{-xy}{2x+2y} = \lambda$$

$$\frac{yz}{y+2z} = \frac{xz}{x+2z}$$

$$(xz)(y+2z) = (yz)(x+2z)$$

$$\cancel{xyz} + 2z^2x = \cancel{xyz} + 2z^2y$$

$$2z^2(x-y) = 0$$

$z = 0$ or: $x - y = 0$

$z = 0$ ↯ disregard $\quad \boxed{x = y}$

$$\frac{xz}{x+2z} = \frac{xy}{2x+2y}$$

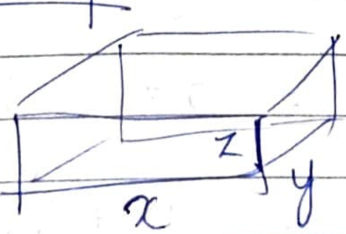
$$(xz)(2x+2y) = (xy)(x+2z)$$

$$2x^2z + \cancel{2xyz} = x^2y + \cancel{2xyz}$$

$$x^2(2z-y) = 0$$

$$\boxed{y = 2z}$$

$\therefore \boxed{x = y = 2z}$ infinite solutions



$\therefore$ the critical points are $(t, t, t/2) \cdot (x, y, z)$
[such that $x = y = 2z$]

(b) $A = 2(xy + yz + zx)$

$V = xyz$

$f(x,y,z) = xyz$
$\phi(x,y,z) = 2(xy + yz + zx)$

Let $F = f + \lambda\phi$.

$F = xyz + 2\lambda(xy + yz + zx)$
At critical point.

$\frac{\partial F}{\partial x} = yz + 2\lambda(y+z) = 0$

$\frac{\partial F}{\partial y} = xz + 2\lambda(x+z) = 0$

$\frac{\partial F}{\partial x} = yx + 2\lambda(y+x) = 0.$

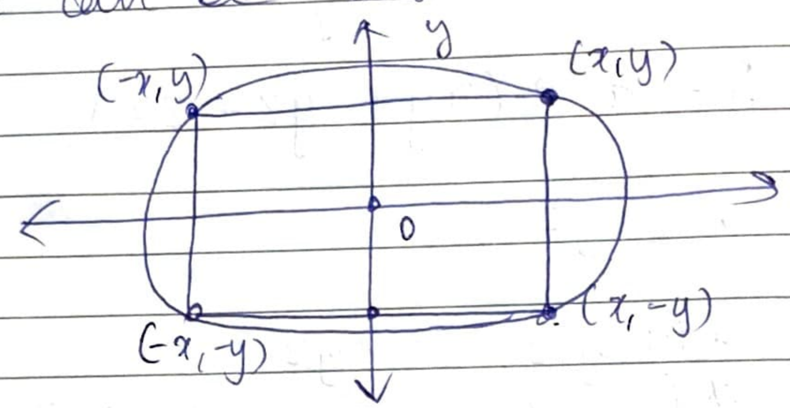
$\frac{+yz}{y+z} = \frac{+xz}{x+z} = \frac{+yx}{y+x} = -2\lambda$

$yz(x+z) = xz(y+z)$
$yz^2 = xz^2.$
$z^2(y-x) = 0 \Rightarrow \boxed{y = x}$

$xz(y+x) = yx(x+z).$
$x^2 z = yx^2$
$x^2(z-y) = 0 \Rightarrow \boxed{z = y}$

$\therefore x = y = z$
$\therefore$ critical points are (x, y, z) where $x = y = z$

28 Find the area of the greatest rectangle that can be inscribed in an ellipse



Ellipse: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Area of rectangle $A = 4xy$

Ellipse : $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \rightarrow (1)$

Let $f(x,y) = 4xy$

$\phi(x,y) = \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Let $F = f + \lambda\phi$

$F = 4xy + \lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)$

$\frac{\partial F}{\partial x} = 4y + \lambda\left(\frac{2x}{a^2}\right) = 0$

$\frac{\partial F}{\partial y} = 4x + \lambda\left(\frac{2y}{b^2}\right) = 0.$

$\frac{-\cancel{4}y \cdot a^2}{\cancel{2}x} = \frac{-\cancel{4}x \cdot b^2}{\cancel{2}y}$

$$\frac{ya^2}{x} = \frac{xb^2}{y} \Rightarrow y^2 a^2 = x^2 b^2$$

$$\Rightarrow \quad y^2 = \frac{x^2 b^2}{a^2}.$$

In eq. (1):

$$\frac{x^2}{a^2} + \frac{x^2 \cancel{b^2}}{\cancel{b^2} a^2} = 1 \Rightarrow \frac{2x^2}{a^2} = 1$$

$$x^2 = \frac{a^2}{2} \Rightarrow \boxed{x = \frac{a}{\sqrt{2}}}$$

$$y^2 = \left(\frac{a^2}{2}\right)\left(\frac{b^2}{a^2}\right) = \frac{b^2}{2} \Rightarrow \boxed{y = \frac{b}{\sqrt{2}}}$$

~~Area of~~ $\therefore$ the critical point is

$$(x, y) = \left(\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}}\right)$$

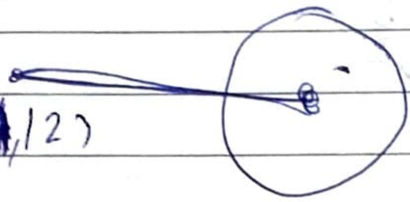
~~$A = 4xy$~~ $\quad f(x,y) = 4xy = \frac{4ab}{2} = 2ab$

$\therefore$ the max. area = ~~2ab units~~ 2ab units

(max: 14, min: 12)
use common sense

1~~8~~6

29. Find the maximum and minimum distances of the point (3, ~~8~~4, 12) from the sphere $x^2+y^2+z^2=1$ (unit sphere).



(3, 4, 12)

Using Lagrange's Method

Let x, y, z be the coordinates of a point on the sphere.

$$\text{Let } f(x,y,z) = \Big(\underbrace{\sqrt{(x-3)^2+(y-4)^2+(z-12)^2}}_{\text{distance}}\Big)^2$$

$$f(x,y,z) = (x-3)^2+(y-4)^2+(z-12)^2$$

$$\phi(x,y,z) = x^2+y^2+z^2=1 = K.$$

Let $F = f + \lambda\phi$

$$F = (x-3)^2 + \cancel{\lambda x^2} + (y-4)^2 + (z-12)^2 + \lambda x^2 + \lambda y^2 + \lambda z^2 = 0$$

$$\frac{\partial F}{\partial x} = \cancel{2}(x-3) + \cancel{2}\lambda x = 0$$

$$\frac{\partial F}{\partial y} = \cancel{2}(y-4) + \cancel{2}\lambda y = 0$$

$$\frac{\partial F}{\partial z} = \cancel{2}(z-12) + \cancel{2}\lambda z = 0$$

$$\frac{x-3}{x} = \frac{y-4}{y} = \frac{z-12}{z} = -\lambda$$

$$\cancel{1}-\frac{3}{x} = \cancel{1}-\frac{4}{y} \qquad \cancel{1}-\frac{4}{y} = \cancel{1}-\frac{12}{z}$$

$$\boxed{3y = 4x} \to (1) \qquad 4z = 12y$$

$$\boxed{z = 3y}$$

$$1-\frac{3}{x} = 1-\frac{12}{z} \Rightarrow 3z = 12x \Rightarrow \boxed{z = 4x}$$

$$y = \frac{4}{3}x \qquad z = 4x$$

170 − 17 = 153

Using $\phi(x, y, z)$.

$$x^2 + \left(\frac{4}{3}x\right)^2 + (4x)^2 = 1$$

153 + 16 = 169

$$x^2 + \frac{16}{9}x^2 + 16x^2 = 1$$

$$17x^2 + \frac{16x^2}{9} = 1$$

$$(153+16)\, x^2 = 9$$

$$x^2 = \frac{9}{169} \Rightarrow \boxed{x = \pm\frac{3}{13}}$$

$$\boxed{y = \pm\frac{4}{13}} \qquad \boxed{z = \pm\frac{12}{13}}$$

$$D_1 = \sqrt{f(x,y,z)} = \sqrt{\left(\frac{3}{13}-3\right)^2 + \left(\frac{4}{13}-4\right)^2 + \left(\frac{12}{13}-12\right)^2}$$

$$= \sqrt{9\left(\frac{12}{13}\right)^2 + 4^2\left(\frac{12}{13}\right)^2 + 12^2\left(\frac{12}{13}\right)^2}$$

$$= \frac{12}{13}\sqrt{9+16+144} = \frac{12}{\cancel{13}} \times \cancel{13} = \boxed{12}$$

$$\boxed{D_1 = 12 \text{ units}}$$

$$D_2 = \sqrt{f(x_2, y_2, z_2)} = \sqrt{\left(\frac{3}{13} - 3\right)^2 + \left(\frac{-4}{13} - 4\right)^2 + \left(\frac{12}{13} - 12\right)^2}$$

$$= \frac{14}{13}\sqrt{3^2 + 4^2 + 12^2} = \frac{14 \times 13}{13}$$

$$\boxed{D_2 = 14 \text{ units}}$$

$\therefore$ the critical values of the distance are 12 units & 14 units.

$\therefore$ min. dist = 12, max dist = 14

Errors & Approximations

(Taylor's Theorem of 2 Variables)

Let $f(x, y)$ be a continuous function of 2 independent variables x and y.

Let δx and δy be the increments in x and y respectively.

Then, the change in the function f can be calculated as

$$\delta f = \frac{\partial f}{\partial x} \cdot \delta x + \frac{\partial f}{\partial y} \cdot \delta y$$

The relative error in f is given by $\frac{\delta f}{f}$

and the percentage error in f is given by $\frac{\delta f}{f} \times 100\%$

30. Find % error in calculating the area of a rectangle when an error of 3% is made in measuring each of its sides.

Let $A = lb$.

$$\delta A = \frac{\partial A}{\partial l} \cdot \delta l + \frac{\partial A}{\partial b} \cdot \delta b.$$

$$\delta A = (b)(\delta l) + l(\delta b)$$

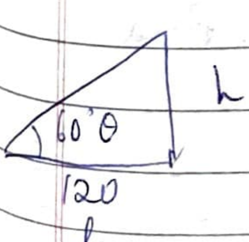
$$\frac{\delta A}{A} = \frac{\delta l}{l} + \frac{\delta b}{b}$$

$\cancel{3.\ 3\%}$ $\quad \frac{\delta A}{A} \times 100\% = \frac{\delta l}{l} \times 100\% + \frac{\delta b}{b} \times 100\%$

$$\frac{\delta A}{A} \times 100\% = 3 + 3 = \boxed{6\%}$$

∴ an error of 6%

31. At a distance 120 ft. from the foot of a tower the elevation of its top is 60°. If the possible errors in measuring the distance and the elevation are 1 inch and 1 min respectively, find the appx. error in the calculated height of the tower.



$$\tan\theta = \frac{h}{l} \Rightarrow h = l\tan\theta$$

$$\delta h = \cancel{\frac{\partial h}{\partial x}}\ \frac{\partial h}{\partial l} \cdot \delta l + \frac{\partial h}{\partial \theta} \cdot \delta\theta$$

170 $120^\circ = \pi$

$\frac{\sec}{\tan}$ $\sec.$ $\frac{\sec}{\sin}$

$h = l \tan\theta$

$\delta h = (\tan\theta)\,\delta l + (l\sec^2\theta)\,\delta\theta$

~~δh~~ when ~~$\theta = 120$~~ $\theta = \pi/3$, $l = 120$ ft.

~~$\frac{\delta h}{h} = \frac{\delta l}{l} + \frac{\sec^2\theta\,\delta\theta}{\tan\theta}$~~

~~$\frac{\delta h}{h} = \frac{\delta l}{l} + \cancel{\cot\delta\theta}\ \frac{\delta\theta}{\sin\theta\cos\theta}$~~

$\delta h = (\tan 60)\left(\frac{1}{12}\right) + (120)(\sec^2 60)\left(\frac{1}{60} \times \frac{\pi}{180}\right)$

$= \frac{\sqrt{3}}{12} + \frac{\cancel{120}^{2} \times 4}{\cancel{60}}\left(\frac{\pi}{180}\right)$

$= \frac{\sqrt{3}}{12} + \frac{8\pi}{180} = \frac{\sqrt{3}}{12} + \frac{2}{45}\pi$

$= 0.144 + 0.139 = \cancel{2.8}\ \cancel{0.284}$

$= 0.284$ feet.

32. The work that must be done to propel a ship of displacement D for a distance S_x in time t is proportional to $\frac{S^2 D^{2/3}}{t^2}$. Find approximately the inc. of work necessary when the displacement is increased by 1%, time is diminished by 1%, the distance is diminished by 2%.

store 67 / /71

Let $W = k \frac{S^2 D^{2/3}}{t^2}$

$\cancel{\delta W = \frac{\partial W}{\partial S}\delta S + \frac{\partial W}{\partial D}\cdot\delta D + \frac{\partial W}{\partial t}\delta t}$

~~δW =~~ Taking ln on both sides.

$\ln W = 2\ln S + \frac{2}{3}\ln D - 2\ln t + \log k.$

$\therefore \frac{\delta W}{W} = \frac{2\delta S}{S} + \frac{2}{3}\frac{\delta D}{D} - \frac{2\delta t}{t}$

$= 2(-0.02) + \frac{2}{3}(0.01) - 2(-0.01)$

$= -0.04 + \frac{0.02}{3} + 0.02 = \frac{0.02 - 0.02}{3}$

$\frac{\delta W}{W} = \frac{-0.04}{3} \Rightarrow -4/3\,\%$

or $\boxed{\frac{\delta W}{W} = -1.33\,\%}$

33. A ballon is in the form of a right circular cylinder of radius 1.5 m and length 4 m and it is surmounted by hemisphere ends. If the radius is increased by 0.01 m and the length by 0.05 m, find % error in the volume of the balloon.

$V = \pi r^2 h + \frac{4}{3}\pi r^3 = \cancel{\pi r^2\left(h + \frac{4}{3}r^3\right)}$

$\delta V = \frac{\partial V}{\partial h}\delta h + \frac{\partial V}{\partial r}\delta r$

$\delta V = \pi r^2 \delta h + (2\pi r h + 4\pi r^2)\delta r$

$\cancel{\frac{\delta V}{V} = \pi r^2}$

1.12 $V = \pi r^2 h + \frac{4}{3}\pi r^3$ $r = 1.5$ $h = 4$

$\delta r = 0.01$ $\delta h = 0.05$

$$\delta V = \pi r^2 \delta h + (2\pi r h + 4\pi r^2)\,\delta r$$

$$\delta V = \pi(2.25)\,\delta h + (2\pi(1.5)(4) + 4\pi(2.25))\,\delta r$$

$$\cancel{V = \pi r^2\left(h + \frac{4}{3}r\right) = \pi r}$$

$$\frac{\delta V}{V} = \frac{\cancel{r^2}\,\delta h}{\cancel{r^2}\left(h + \frac{4}{3}r\right)} + \frac{(2hr + 4\times r^2)\,\delta r}{r^2\left(h + \frac{4}{3}r\right)}$$

$$\frac{\delta V}{V} = \frac{\delta h}{h + \frac{4r}{3}} + \frac{(2h + 4r)\,\delta r}{r\left(h + \frac{4r}{3}\right)}$$

$$= \frac{0.01}{(4+2)} + \frac{(8+6)\,0.05}{1.5(4+2)}$$

$$\frac{\delta V}{V} = \frac{0.01}{6} + \frac{(14)(0.05)}{(1.5)(6)}$$

$$= \frac{1}{600} + \frac{7}{90}$$

$$\frac{\delta V}{V} = \frac{\cancel{\pi r^2}\,\delta h}{\cancel{\pi r^2}\left(h + \frac{4}{3}r\right)} + \frac{2\cancel{\pi r}(h + 2r)\,\delta r}{\cancel{\pi} r^2\left(h + \frac{4}{3}r\right)}$$

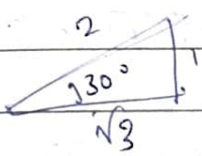
$$\frac{\delta V}{V} = \frac{\delta h}{\left(h + \frac{4}{3}r\right)} + \frac{2(h + 2r)\,\delta r}{r\left(h + \frac{4}{3}r\right)}$$

$$= \frac{0.05}{(4+2)} + \frac{2(4+3)\,\delta r}{1.5(4+2)}$$

$$= \frac{0.05}{6} + \frac{2\times 7\,\delta r}{9}$$

$$\frac{\delta V}{V} = \frac{1}{120} + \frac{1}{450} = 2.389\%$$

14. The height h, and the semivertical angle α of a right circular cone are measured and using them the total surface area of the cone is calculated. If h and α are both in error by small quantities Δh and $\Delta\alpha$, find the correspondg error in the area.
Show further that if $\alpha = \pi/6$, an error of 1% in h will ~~Δα~~ be approximately compensated by an error of $-0.33°$ in α.



$\tan\alpha = \dfrac{r}{h}$ $\qquad \cos\alpha = \dfrac{h}{l}$

$r = h\tan\alpha$

$l = h\sec\alpha$

$$A = \pi r l + \pi r^2 = \pi r(l + r)$$
$$= \pi h\tan\alpha(h\sec\alpha + h\tan\alpha)$$
$$A = \pi h^2(\sec\alpha\tan\alpha + \tan^2\alpha)$$

$$\Delta A = \frac{\partial A}{\partial h}\Delta h + \frac{\partial A}{\partial\alpha}\Delta\alpha$$

$$\boxed{\Delta A = 2\pi h(\sec\alpha\tan\alpha + \tan^2\alpha)\Delta h + \pi h^2(\sec\alpha\tan^2\alpha + \sec^3\alpha + 2\tan\alpha\sec^2\alpha)\Delta\alpha}$$

If $\alpha = \pi/6$

~~$\Delta A = 2\pi h$~~

$$\Delta A = 2\pi h\left(\frac{2}{\sqrt3}\cdot\frac{1}{\sqrt3} + \frac{1}{3}\right)\Delta h + \pi h^2\left(\frac{2}{\sqrt3}\cdot\frac{1}{3} + \frac{8}{3\sqrt3} + \frac{2\cdot4}{\sqrt3\,3}\right)\Delta\alpha$$

$$\Delta A = 2\pi h(1)\Delta h + \pi h^2\left(\frac{2}{3\sqrt{3}} + \frac{8}{3\sqrt{3}} + \frac{8}{3\sqrt{3}}\right)\Delta\alpha$$

$$\Delta A = 2\pi h\Delta h + \pi h^2\left(\frac{6\sqrt{3}}{\sqrt{3}\sqrt{3}}\right)\Delta\alpha$$

$$\Delta A = 2\pi h\Delta h + \pi h^2(2\sqrt{3})\Delta\alpha$$

$$\frac{\Delta h}{h} = 0.01 \Rightarrow \Delta h = 0.01h.$$

$$\Delta\alpha = \frac{-0.33\pi}{180}.$$

$$\Delta A = 2\pi h^2(0.01) + \pi h^2(2\sqrt{3})\Delta\alpha = 0.$$

$$0.02\cancel{\pi h^2} = -\cancel{\pi h^2}(2\sqrt{3})\Delta\alpha$$

$$\Delta\alpha = \frac{-0.02}{2\sqrt{3}} = -5.77\times10^{-3}$$

$$\Delta\alpha \text{ in degrees} = \frac{180\times -5.77\times10^{-3}}{\pi}$$

$$\Delta\alpha = 0.33^\circ.$$

85.